Jason Ferguson

Solutions to Homework Assignment #2 for MATH 54.002

My solutions may not be the best way to solve these problems, and in fact you may find solutions that are better than mine. Congratulations to you if you do!

Section 1.3

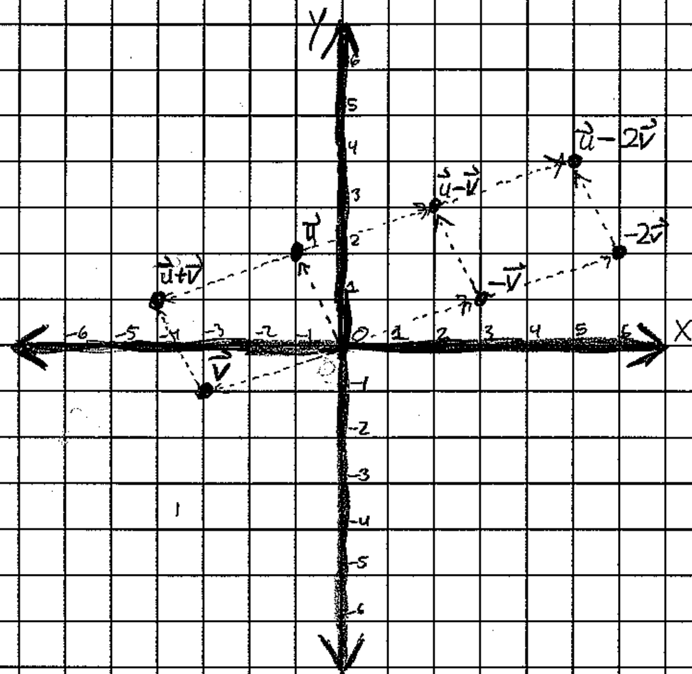
1. $\vec{u}+\vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \boxed{\begin{bmatrix} -4 \\ 1 \end{bmatrix}}$

   $\vec{u}-2\vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} - \begin{bmatrix} -6 \\ -2 \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ 4 \end{bmatrix}}$

3. See back.

Jason Ferguson

3.



Jason Ferguson

5. $x_1 \begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$ is $\begin{bmatrix} 6x_1 - 3x_2 \\ -x_1 + 4x_2 \\ 5x_1 \end{bmatrix}$.

Answer is $\boxed{\begin{aligned} 6x_1 - 3x_2 &= 1 \\ -x_1 + 4x_2 &= -7 \\ 5x_1 \quad &= -5 \end{aligned}}$.

7. $\vec{a} = \boxed{\vec{u} - 2\vec{v}}$.

$\vec{b} = \vec{a} + \vec{u} = \boxed{2\vec{u} - 2\vec{v}}$

$\vec{c} = \vec{b} + \frac{3}{2}(-\vec{v}) = \left(2\vec{u} - \frac{4}{2}\vec{v}\right) - \frac{3}{2}\vec{v} = \boxed{2\vec{u} - \frac{7}{2}\vec{v}}$

$\vec{d} = \vec{b} + \vec{a} - 2\vec{v} = \boxed{3\vec{u} - 4\vec{v}}$

Every point of $\mathbb{R}^2$ is on the intersection of a line parallel to $\vec{u}$ and a line parallel to $\vec{v}$. This means $\boxed{\text{every vector in } \mathbb{R}^2 \text{ is a linear combination of } \vec{u} \text{ and } \vec{v}.}$

9. $\boxed{x_1 \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$.

11. $\vec{b}$ is a linear combination of $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$ only when $\vec{a}_1x_1 + \vec{a}_2x_2 + \vec{a}_3x_3 = \vec{b}$ has a solution, so we reduce the augmented matrix $[\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \ \vec{b}]$ to check this.

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix} \xrightarrow[\text{to Row 2}]{\text{Add } 2\times(\text{Row 1})} \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{bmatrix}.$$

$$\xrightarrow[\text{to Row 3}]{\text{Add } (-2)\times(\text{Row 2})} \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The system is consistent, so $\boxed{\vec{b} \text{ is a linear combination of } \vec{a}_1, \vec{a}_2, \text{ and } \vec{a}_3.}$

Jason Ferguson

13. $\vec{b}$ is a linear combination of the columns of A only when there are numbers x_1, x_2, and x_3 for which

$$x_1\begin{bmatrix}1\\0\\-2\end{bmatrix}+x_2\begin{bmatrix}-4\\3\\8\end{bmatrix}+x_3\begin{bmatrix}2\\5\\-4\end{bmatrix}=\begin{bmatrix}3\\-7\\-3\end{bmatrix}.$$

This happens only when there is a solution (x_1,x_2,x_3) ~~to~~ the system

$$\begin{aligned}x_1-4x_2+2x_3&=3\\3x_2+5x_3&=-7\\-2x_1+8x_2-4x_3&=-3.\end{aligned}$$

So we check this system for consistency

$$\begin{bmatrix}1&-4&2&3\\0&3&5&-7\\-2&8&-4&-3\end{bmatrix}\xrightarrow[\text{to Row 3}]{\text{Add }2\times(\text{Row }1)}\begin{bmatrix}1&-4&2&3\\0&3&5&-7\\0&0&0&3\end{bmatrix}.$$

The system has $[0,0\ 0\ 3]$ as a row, so inconsistent, so $\boxed{\vec{b}\text{ is not a linear combination of the columns of }A.}$

15. $\boxed{0\vec{v}_1+0\vec{v}_2}=\begin{bmatrix}0\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\0\end{bmatrix}=\boxed{\begin{bmatrix}0\\0\\0\end{bmatrix}}$

$\boxed{1\vec{v}_1+0\vec{v}_2}=\begin{bmatrix}7\\1\\-6\end{bmatrix}+\begin{bmatrix}0\\0\\0\end{bmatrix}=\boxed{\begin{bmatrix}7\\1\\-6\end{bmatrix}}$

$\boxed{0\vec{v}_1+1\vec{v}_2}=\begin{bmatrix}0\\0\\0\end{bmatrix}+\begin{bmatrix}-5\\3\\0\end{bmatrix}=\boxed{\begin{bmatrix}-5\\3\\0\end{bmatrix}}$

$\boxed{1\vec{v}_1+1\vec{v}_2}=\begin{bmatrix}7\\1\\-6\end{bmatrix}+\begin{bmatrix}-5\\3\\0\end{bmatrix}=\boxed{\begin{bmatrix}2\\4\\-6\end{bmatrix}}$

$\boxed{(-1)\vec{v}_1+0\vec{v}_2}=-\vec{v}_1=\boxed{\begin{bmatrix}-7\\-1\\6\end{bmatrix}}.$

(There are many other possible choices.)

22. $\boxed{\text{Let }A=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\text{ and }\vec{b}=\begin{bmatrix}0\\0\\1\end{bmatrix}.}$ Then $\vec{b}$ is in the span of the columns of A only when the augmented matrix $[A\ \vec{b}]$ corresponds to a consistent system (see my solution to 1.3.13). But

$$[A\,\vec{b}]=\begin{bmatrix}1&1&1&0\\1&1&1&0\\1&1&1&1\end{bmatrix}\xrightarrow[\text{to Row 3}]{\text{Add }(-1)\times(\text{Row }1)}\begin{bmatrix}1&1&1&0\\1&1&1&0\\0&0&0&1\end{bmatrix},$$

so $[A\,\vec{b}]$ corresponds to an inconsistent system.

25. (a). $\vec{b}$ is not either $\vec{a}_1$, $\vec{a}_2$, or $\vec{e}_3$, so $\boxed{\vec{b} \text{ is not in } \{\vec{a}_1, \vec{a}_2, \vec{a}_3\}.}$

Because $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$ are all different, $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ has $\boxed{\text{three}}$ elements.

(b) We check if $[A \ \vec{b}]$ is consistent. (See my answer to 1.3.13 to see why.)

$$\begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ -2 & 6 & 3 & -4 \end{bmatrix} \xrightarrow[\text{to Row 3}]{\text{Add } 2\times(\text{Row 1})} \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 6 & -5 & 4 \end{bmatrix}$$

$$\xrightarrow[\text{to Row 3}]{\text{Add}(-2)\times(\text{Row 2})} \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 2 \\ 0 & 0 & -1 & 6 \end{bmatrix}.$$

This is the augmented matrix of a consistent system, so $\boxed{\vec{b} \text{ is in } W.}$

$\boxed{\text{W has infinitely many elements.}}$ For example, $r\vec{a}_1$ is in W for every real number r, and if $r_1 \neq r_2$, then $r_1\vec{a}_1 \neq r_2\vec{a}_1$. Because there are infinitely many real numbers, W has infinitely elements.

(c) Because $\vec{a}_1 = 1\cdot\vec{a}_1 + 0\cdot\vec{a}_2 + 0\cdot\vec{a}_3$, $\vec{a}_1$ is a linear combination of $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$, so $\vec{a}_1$ is in W.

Section 1.4

1. The product is $\boxed{\text{undefined}}$ since the first matrix has two columns and the second matrix has three rows.

5. $$\boxed{5\begin{bmatrix} 5 \\ -2 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ -7 \end{bmatrix} + 3\begin{bmatrix} -8 \\ 3 \end{bmatrix} + (-2)\begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}}$$

7. $$\boxed{\begin{bmatrix} 4 & -5 & 7 \\ -1 & 3 & -8 \\ 7 & -5 & 0 \\ -4 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}}.$$

9. $$\boxed{x_1\begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}} \qquad \boxed{\begin{bmatrix} 3 & 1 & -5 \\ 0 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}}$$

Jason Ferguson

11. Augmented matrix is $\boxed{\begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ -2 & -4 & -3 & 9 \end{bmatrix}}$.

$$\xrightarrow{\text{Add } 2\times(\text{Row }1) \text{ to Row } 3} \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 5 & 5 \end{bmatrix} \xrightarrow[\text{by } \frac{1}{5}]{\text{Multiply Row 3}} \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[\text{Add } (-5)\times(\text{Row }3) \text{ to } (\text{Row }2)]{\text{Add } (-4)\times(\text{Row }3) \text{ to } (\text{Row }1)} \begin{bmatrix} 1 & 2 & 0 & -6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[\text{to Row 1}]{\text{Add } (-2)\times(\text{Row }2)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Solution is $\boxed{\begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}}$.

17. $$\begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} \xrightarrow[\text{Add } (-2)\times(\text{Row }1) \text{ to Row } 4]{\text{Add Row 1 to Row 2}} \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & -4 & 2 & -8 \\ 0 & -6 & 3 & -7 \end{bmatrix}$$

$$\xrightarrow[\text{Add } 3\times(\text{Row }2) \text{ to Row } 4]{\text{Add } 2\times(\text{Row }2) \text{ to Row } 3} \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$\xrightarrow[\text{Rows 3 and 4}]{\text{Exchange}} \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\text{Three}}$ rows of A have a pivot position.

$\boxed{\text{By theorem 4}}$, $\boxed{A\vec{x} = \vec{b} \text{ does not have a solution for all } \vec{b} \text{ in } \mathbb{R}^4.}$

18. $$\begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix} \xrightarrow[\text{Add } 2\times(\text{Row }1) \text{ to Row } 4]{\text{Add } (-1)\times(\text{Row }1) \text{ to Row } 3} \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & -1 & -1 & 5 \\ 0 & -2 & -2 & 3 \end{bmatrix}$$

$$\xrightarrow[\text{Add } 2(\text{Row }2) \text{ to Row } 4]{\text{Add Row 2 to Row 3}} \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix} \xrightarrow[\text{and 4}]{\text{Exchange Rows 3}} \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Only three rows of B have a pivot position. $\boxed{\text{By theorem 4}}$, $\boxed{\text{the columns of } B \text{ do not span } \mathbb{R}^4}$, and $\boxed{B\vec{x} = \vec{y} \text{ does not have a solution for every } \vec{y} \text{ in } \mathbb{R}^4.}$

29. Let $\boxed{A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}$. Then A is not in echelon form. But,

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow[\text{1 and 3}]{\text{Exchange Rows}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so A has a pivot position in every row, and therefore, the columns of A span $\mathbb{R}^3$.

(There are many, many other examples.)

34. Because $A\vec{x} = \vec{b}$ has a unique solution, the corresponding system of equations has no free variables and so has three basic variables. Therefore, the augmented matrix $[A\ \vec{b}]$ has three pivot positions, in its ~~three left columns~~. Then A also has three pivot positions, and so has a pivot position in every row.

Therefore, the columns of A span $\mathbb{R}^3$ by Theorem 4, as needed.

Section 1.5

1. $$\begin{bmatrix} 2 & -5 & 8 & 0 \\ -2 & -7 & 1 & 0 \\ 4 & 2 & 7 & 0 \end{bmatrix} \xrightarrow[\text{Add }(-2)\times(\text{Row }1)\text{ to Row }3]{\text{Add Row 1 to Row 2}} \begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 12 & -9 & 0 \end{bmatrix}$$

$$\xrightarrow[\text{to Row 3}]{\text{Add Row 2}} \begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The system has a free variable, so it $\boxed{\text{has a nontrivial solution.}}$

5. $$\begin{bmatrix} 1 & 3 & 1 & 0 \\ -4 & -9 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \xrightarrow[\text{to Row 2}]{\text{Add }4\times(\text{Row }1)} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix}$$

$$\xrightarrow[\text{to Row 3}]{\text{Add Row 2}} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{by }1/3]{\text{Multiply Row 2}} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[\text{to Row 1}]{\text{Add }(-3)(\text{Row }2)} \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

General solution is $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} x_3 \text{ for any } x_3 \text{ in } \mathbb{R}}$

Jason Ferguson

9. $\begin{bmatrix} 3 & -9 & 6 \\ -1 & 3 & -2 \end{bmatrix} \xrightarrow[\text{to Row 2}]{\text{Add } \frac{1}{3}\times(\text{Row 1})} \begin{bmatrix} 3 & -9 & 6 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{by } \frac{1}{3}]{\text{Multiply Row 1}} \begin{bmatrix} 1 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

Solutions are $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \text{ for all } x_2, x_3 \text{ in } \mathbb{R}.}$

14. Solution set is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_4 \\ 8+x_4 \\ 2-5x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 3 \\ 1 \\ -5 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 8 \\ 2 \\ 0 \end{bmatrix}.$$

This is $\boxed{\text{the line through } \begin{bmatrix} 3 \\ 8 \\ 2 \\ 0 \end{bmatrix} \text{ parallel to the vector } \begin{bmatrix} 3 \\ 1 \\ -5 \\ 1 \end{bmatrix}}$.

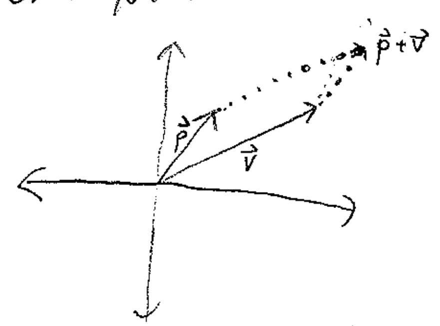
24a. $\boxed{\text{False}}$ You only need one entry of $\vec{x}$ to be nonzero. For example, since $\begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [1 \cdot 0 + 0 \cdot 1] = [0]$,

so $\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a nontrivial solution to $[1\ 0] \cdot \vec{x} = \vec{0}$.

b. $\boxed{\text{True}}$. Any equation of the form $\vec{x} = \vec{p} + x_2\vec{u} + x_3\vec{v}$, where neither $\vec{u}$ nor $\vec{v}$ are multiples of each other, is a plane passing through $\vec{p}$. The equation $\vec{x} = x_2\vec{u} + x_3\vec{v}$ corresponds to $\vec{p} = \vec{0}$, so it's a plane through the origin.

c. $\boxed{\text{True}}$. A homogeneous system, by definition, is a system of the form $A\vec{x} = \vec{0}$ for some matrix A. $A\vec{x} = \vec{b}$ doesn't look like $A\vec{x} = \vec{0}$ at first glance, but since you know that $\vec{x} = 0$ is a solution, you know $A \cdot \vec{0} = \vec{b}$. But $A \cdot \vec{0} = \vec{0}$, so you know $\vec{b} = \vec{0}$.

d. $\boxed{\text{True}}$



e. $\boxed{\text{False}}$ This only holds when $A\vec{x} = \vec{b}$ has at least one solution. For example, the equation $0x = 0$ has every real number as a solution, but $0x = 1$ has no solutions, so the solution set of $0x = 1$ isn't obtained by translating the solution set ~~of~~ $0x = 0$.

29(a). $A\vec{x}=\vec{0}$ has a nontrivial solution only when A has a free variable. Since A has three columns and three pivot positions, A has no free variables, so $A\vec{x}=\vec{0}$ $\boxed{\text{does not have a nontrivial solution.}}$

(b) Since A has three pivot positions and three rows, A has a pivot position in every row, so $\boxed{A\vec{x}=\vec{b} \text{ has a solution for every } \vec{b}.}$

30(a) Since A has three columns and two pivot positions, A has a free variable, so $\boxed{A\vec{x}=\vec{0} \text{ has nontrivial solutions.}}$

(b) Since A has three rows but only two pivot positions, A does not have a pivot position in every row, so $\boxed{A\vec{x}=\vec{b} \text{ does not have a solution for every } \vec{b}.}$

31(a). Since A has two pivot positions and two columns A has no free variables, so $A\vec{x}=\vec{0}$ $\boxed{\text{does not have a nontrivial solution.}}$

(b) Since A has two pivot positions and three rows, A does not have a pivot position in every row, so $\boxed{A\vec{x}=\vec{b} \text{ does not have a solution for every } \vec{b}.}$

32(a) A has two pivot positions and four columns, so A does not have a pivot position in every column. A has free variables, so $\boxed{A\vec{x}=\vec{0} \text{ has a nontrivial solution.}}$

(b) A has two pivot positions and two rows, so A has a pivot position in every row, so $\boxed{A\vec{x}=\vec{b} \text{ has a solution for every } \vec{b}.}$

Section 1.6

1. Let g and s be the outputs of goods and services at equilibrium. Then output of goods is its input, so

$$g = 0.2g + 0.7s$$

Output of services is its inputs so

$$s = 0.3s + 0.8g$$

Can rearrange to get: $-0.8g + 0.7s = 0$
$-0.8g + 0.7s = 0.$

So $s = \frac{0.7}{0.8}g = 0.875g.$ $\boxed{\text{Output of services is } 0.875 \text{ times output of goods.}}$

Jason Ferguson

3. "c" is the equilibrium output of Chemicals & Metals,
"f" is " " " of Fuels and Power
"m" is " " " of Machinery

(a)

Output c	Output f	Output m	
0.2	0.8	0.4	c purchased by
0.3	0.1	0.4	f purchased by
0.5	0.1	0.2	m purchased by

(b) At equilibrium, each company's output is equal to its purchases:

$$c = 0.2c + 0.8f + 0.4m = \tfrac{1}{5}c + \tfrac{4}{5}f + \tfrac{2}{5}m$$
$$f = 0.3c + 0.1f + 0.4m = \tfrac{3}{10}c + \tfrac{1}{10}f + \tfrac{2}{5}m$$
$$m = 0.5c + 0.1f + 0.2m = \tfrac{1}{2}c + \tfrac{1}{10}f + \tfrac{1}{5}m.$$

Rearranging gives:

$$\boxed{\begin{aligned} -\tfrac{4}{5}c + \tfrac{4}{5}f + \tfrac{2}{5}m &= 0 \\ \tfrac{3}{10}c - \tfrac{9}{10}f + \tfrac{2}{5}m &= 0 \\ \tfrac{1}{2}c + \tfrac{1}{10}f - \tfrac{4}{5}m &= 0 \end{aligned}}$$

Augmented matrix is:

$$\begin{bmatrix} -\frac{4}{5} & \frac{4}{5} & \frac{2}{5} & 0 \\ \frac{3}{10} & -\frac{9}{10} & \frac{2}{5} & 0 \\ \frac{1}{2} & \frac{1}{10} & -\frac{4}{5} & 0 \end{bmatrix}$$

(Your equations and matrix might look different, depending on whether you converted to fractions or not, and what your algebraic steps were.)

(c) Multiply Row 1 by $(-\frac{5}{4})$
Multiply Row 2 by 10
Multiply Row 3 by 10

$$\longrightarrow \begin{bmatrix} 1 & -1 & -\frac{1}{2} & 0 \\ 3 & -9 & 4 & 0 \\ 5 & 1 & -8 & 0 \end{bmatrix}$$

Add $(-3)\times$(Row 1) to Row 2
Add $(-5)\times$(Row 1) to Row 3

$$\longrightarrow \begin{bmatrix} 1 & -1 & -\frac{1}{2} & 0 \\ 0 & -6 & \frac{11}{2} & 0 \\ 0 & 6 & -\frac{11}{2} & 0 \end{bmatrix}$$

$(4 + \frac{3}{2} = \frac{11}{2})$
$(-8 + \frac{5}{2} = -\frac{11}{2})$

$$\xrightarrow{\text{Add Row 2 to Row 3}} \begin{bmatrix} 1 & -1 & -\frac{1}{2} & 0 \\ 0 & -6 & \frac{11}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Multiply Row 2 by } -\frac{1}{6}} \begin{bmatrix} 1 & -1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{11}{12} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(continued on next page)

(Copied from previous page)
$$\left[\begin{array}{ccc|c} 1 & -1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{11}{12} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\text{to Row 1}]{\text{Add Row 2}} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{17}{12} & 0 \\ 0 & 1 & -\frac{11}{12} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$c = \frac{17}{12}m$, $f = \frac{11}{12}m$, m is free is the solution to the equation.
But the problem says $m = 100$, so:

$$\boxed{(c, f, m) = \left(\frac{1700}{12}, \frac{1100}{12}, 100\right) \approx (140, 92, 100)}$$

9. Let x_1 through x_6 be the numbers that should appear in front of the first through sixth chemicals in the reaction. Then:

from Pb: $x_1 = 3x_3$

from N: $6x_1 = x_6$

from Cr: $x_2 = 2x_4$

from Mn: $2x_2 = x_5$

from O: $8x_2 = 4x_3 + 3x_4 + 2x_5 + x_6$

Rearrange to get:

$$\begin{array}{llllllll} x_1 & & -3x_3 & & & & = 0 \\ 6x_1 & & & & & -x_6 & = 0 \\ & x_2 & & -2x_4 & & & = 0 \\ & 2x_2 & & & -x_5 & & = 0 \\ & 8x_2 & -4x_3 & -3x_4 & -2x_5 & -x_6 & = 0 \end{array}$$

Convert into an augmented matrix:

$$\left[\begin{array}{cccccc|c} 1 & 0 & -3 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 8 & -4 & -3 & -2 & -1 & 0 \end{array}\right] \xrightarrow[\text{to Row 2}]{\text{Add }(-6)\times(\text{Row 1})} \left[\begin{array}{cccccc|c} 1 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 18 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 8 & -4 & -3 & -2 & -1 & 0 \end{array}\right]$$

(Continued on back)

Recopied
$$\xrightarrow{\text{Recopied}} \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 18 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 8 & -4 & -3 & -2 & -1 & 0 \end{bmatrix} \xrightarrow{\text{Switch Rows 2 and 3}} \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 18 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 8 & -4 & -3 & -2 & -1 & 0 \end{bmatrix}$$

Add (-2)x(Row2) to Row4
Add (-8)x(Row2) to Row5
$$\longrightarrow \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 18 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 & -1 & 0 & 0 \\ 0 & 0 & -4 & 13 & -2 & -1 & 0 \end{bmatrix} \xrightarrow{\text{Add } \left(\frac{2}{9}\right)\times(\text{Row3}) \text{ to Row 5}} \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 18 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 13 & -2 & -\frac{11}{9} & 0 \end{bmatrix}$$

Add $\left(-\frac{13}{4}\right)\times$(Row 4) to Row 5
$$\longrightarrow \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 18 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{4} & -\frac{11}{9} & 0 \end{bmatrix}$$

Multiply Row 3 by $\frac{1}{18}$
Multiply Row 4 by $\frac{1}{4}$
Multiply Row 5 by $\frac{4}{5}$
$$\longrightarrow \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{18} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{44}{45} & 0 \end{bmatrix}$$

Add $\left(\frac{1}{4}\right)\times$(Row 5) to Row 4
$$\longrightarrow \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{18} & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{11}{45} & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{44}{45} & 0 \end{bmatrix} \xrightarrow{\text{Add } 2\times(\text{Row4}) \text{ to Row2}} \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -22/45 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1/18 & 0 \\ 0 & 0 & 0 & 1 & 0 & -11/45 & 0 \\ 0 & 0 & 0 & 0 & 1 & -44/45 & 0 \end{bmatrix}$$

Add $3\times$(Row 3) to Row 1
$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1/6 & 0 \\ 0 & 1 & 0 & 0 & 0 & -22/45 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1/18 & 0 \\ 0 & 0 & 0 & 1 & 0 & -11/45 & 0 \\ 0 & 0 & 0 & 0 & 1 & -44/45 & 0 \end{bmatrix}.$$

So $(x_1, x_2, x_3, x_4, x_5, x_6) = \left(\frac{1}{6}x_6, \frac{22}{45}x_6 \ \frac{1}{18}x_6, \frac{11}{45}x_6, \frac{44}{45}x_6, x_6\right)$.

Choosing $x_6 = 90$ to get all of x_1 through x_6 to be integers gives

$$\boxed{15\,PbN_6 + 44\,CrMn_2O_8 \longrightarrow 5\,Pb_3O_4 + 22\,Cr_2O_3 + 88\,MnO_2 + 90\,NO}$$

11: From intersection A: $x_1 + x_3 = 20$
From intersection B: $x_2 = x_3 + x_4$
From intersection C: $80 = x_1 + x_2$
From outer edges: $80 = 20 + x_4$.

Rearrange:
$$\begin{aligned} x_1 \quad\quad + x_3 \quad\quad &= 20 \\ x_2 - x_3 - x_4 &= 0 \\ x_1 + x_2 \quad\quad\quad\quad &= 80 \\ x_4 &= 60 \end{aligned}$$

(Continued on next page)

Augmented Matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 20 \\ 0 & 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 & 80 \\ 0 & 0 & 0 & 1 & 60 \end{bmatrix} \xrightarrow[\text{to Row3}]{\text{Add }(-1)\times(\text{Row}1)} \begin{bmatrix} 1 & 0 & 1 & 0 & 20 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 60 \\ 0 & 0 & 0 & 1 & 60 \end{bmatrix}$$

$$\xrightarrow[\text{to Row3}]{\text{Add }(-1)\times(\text{Row}2)} \begin{bmatrix} 1 & 0 & 1 & 0 & 20 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 60 \\ 0 & 0 & 0 & 1 & 60 \end{bmatrix} \xrightarrow[\text{Add }(-1)\times\text{Row3 to Row4}]{\text{Add Row3 to Row2}} \begin{bmatrix} 1 & 0 & 1 & 0 & 20 \\ 0 & 1 & -1 & 0 & 60 \\ 0 & 0 & 0 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

Solution is

$$\boxed{\begin{aligned} x_1 &= 20 - x_3 \\ x_2 &= 60 + x_3 \\ x_3 &\text{ is free} \\ x_4 &= 60 \end{aligned}}$$

If $x_1 = 20 - x_3$ is nonnegative then $x_3 \le 20$. $x_3 = 20$ works since you get $(x_1, x_2, x_3, x_4) = (0, 80, 20, 60)$. So $\boxed{20}$ is the largest x_3 can be.

14. Intersection A: $x_1 = x_2 + 100$

" B: $50 + x_2 = x_3$

" C: $x_3 = x_4 + 120$

" D: $x_4 + 150 = x_5$

" E: $x_5 = x_6 + 80$

" F: $x_6 + 100 = x_1$

Outside: $50 + 150 + 100 = 100 + 120 + 80$

Rearrange:

$$\begin{aligned} x_1 - x_2 &= 100 \\ x_2 - x_3 &= -50 \\ x_3 - x_4 &= 120 \\ x_4 - x_5 &= -150 \\ x_5 - x_6 &= 80 \\ x_1 \qquad\qquad - x_6 &= 100 \end{aligned}$$

(Continued on back)

Augmented matrix is:

$$\left[\begin{array}{cccccc|c} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 1 & 0 & 0 & 0 & 0 & -1 & 100 \end{array}\right] \xrightarrow[\text{to Row6}]{\text{Add}(-1)\times(\text{Row1})} \left[\begin{array}{cccccc|c} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[\text{to Row6}]{\text{Add}(-1)\times(\text{Row2})} \left[\begin{array}{cccccc|c} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 1 & 0 & 0 & -1 & 50 \end{array}\right] \xrightarrow[\text{to Row6}]{\text{Add}(-1)\times(\text{Row3})} \left[\begin{array}{cccccc|c} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \end{array}\right]$$

$$\xrightarrow[\text{to Row6}]{\text{Add}(-1)\times(\text{Row4})} \left[\begin{array}{cccccc|c} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \end{array}\right] \xrightarrow[\text{to Row6}]{\text{Add}(-1)\times(\text{Row5})} \left[\begin{array}{cccccc|c} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

So:

$$\boxed{\begin{aligned} &x_6 && \text{is free} \\ &x_5 && = x_6 + 80 \\ &x_4 = x_5 - 150 && = x_6 - 70 \\ &x_3 = x_4 + 120 && = x_6 + 50 \\ &x_2 = x_3 - 50 && = x_6 \\ &x_1 = x_2 + 100 && = x_6 + 100 \end{aligned}}$$

Since all roads are one-way, none of x_1 through x_6 can be negative. So $x_4 = x_6 - 70 \geq 0$, so $x_6 \geq 70$. If $x_6 = 70$, then $(x_1, \dots, x_6) = (170, 70, 120, 0, 150, 70)$, so $\boxed{70}$ is the smallest x_6 can be.

Section 1.7

1. Must row-reduce $\begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & -6 & -8 \end{bmatrix}$; if the row-reduced matrix has a pivot in every column, then the vectors are linearly independent; otherwise, they are dependent.

$$\begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & -6 & -8 \end{bmatrix} \xrightarrow[\text{to Row3}]{\text{Add } 3\times(\text{Row2})} \begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}.$$

The row-reduced matrix has a pivot in every column, so the vectors are $\boxed{\text{linearly independent.}}$

Jason Ferguson

3. $\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} \xrightarrow[\text{to Row2}]{\text{Add } 3\times(\text{Row1})} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$.

There is not a pivot in every column, so the vectors are $\boxed{\text{not linearly independent}}$

5. $\begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & -3 & 2 \end{bmatrix} \xrightarrow[\text{1 and 4}]{\text{Switch Rows}} \begin{bmatrix} 1 & -3 & 2 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 0 & -8 & 5 \end{bmatrix} \xrightarrow[\text{Add Row1 to Row3}]{\text{Add } (-3)\times\text{Row1 to Row2}} \begin{bmatrix} 1 & -3 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & -2 \\ 0 & -8 & 5 \end{bmatrix}$

$\xrightarrow[\text{Add } 4\times(\text{Row2}) \text{ to Row4}]{\text{Add } (-1)\times\text{Row2 to Row3}} \begin{bmatrix} 1 & -3 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow[\text{and 4}]{\text{Switch Rows 3}} \begin{bmatrix} 1 & -3 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$.

There is a pivot in every column, so the columns are $\boxed{\text{linearly independent.}}$

7. There are more columns than entries in each column, so the columns are $\boxed{\text{not linearly independent}}$

9(a) $\vec{v}_3$ is in Span$\{\vec{v}_1, \vec{v}_2\}$ if and only if the equation $[\vec{v}_1 \ \vec{v}_2]\cdot\vec{x} = \vec{v}_3$ has a solution $\vec{x}$, i.e. the following augmented matrix is consistent

$\begin{bmatrix} 1 & -3 & 5 \\ -3 & 9 & -7 \\ 2 & -6 & h \end{bmatrix} \xrightarrow[\text{Add } (-2)\times(\text{Row1}) \text{ to Row3}]{\text{Add } 3\times(\text{Row1}) \text{ to Row2}} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & h-10 \end{bmatrix}$

Because of the row $[0 \ \ 0 \ \ 8]$, the system is always inconsistent, so $\boxed{\vec{v}_3 \text{ is not in Span}\{\vec{v}_1, \vec{v}_2\}}$ $\boxed{\text{for any value } h.}$

(b) $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent only when the matrix $[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$ does not have a pivot in every column. But from the calculations in part (a), the matrix never has a pivot in column 2, so $\boxed{\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \text{ is dependent for all } h.}$

Also, $\vec{v}_2 = -3\vec{v}_1$, so $3\vec{v}_1 + 1\vec{v}_2 + 0\vec{v}_3 = \vec{0}$, so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is dependent.

11. $\begin{bmatrix} 1 & 3 & -1 \\ -1 & -5 & 5 \\ 4 & 7 & h \end{bmatrix} \xrightarrow[\text{Add } (-4)\times\text{Row1 to Row3}]{\text{Add Row1 to Row2}} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -2 & 4 \\ 0 & -5 & h+4 \end{bmatrix} \xrightarrow[\text{by } -\frac{1}{2}]{\text{Multiply Row2}} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ 0 & -5 & h+4 \end{bmatrix}$

$\xrightarrow[\text{to Row2}]{\text{Add } 5\times(\text{Row3})} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & h-6 \end{bmatrix}$

This ~~has~~ has a pivot in every column only when $h \neq 6$, so the vectors are dependent $\boxed{\text{only when } h=6.}$

13. $\begin{bmatrix} 1 & -2 & 3 \\ 5 & -9 & h \\ -3 & 6 & -9 \end{bmatrix} \xrightarrow[\text{Add } 3\times \text{Row 1 to Row 3}]{\text{Add } (-5)\times \text{Row 1 to Row 2}} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & h-15 \\ 0 & 0 & 0 \end{bmatrix}$.

This matrix never has a pivot in column 3, so the vectors are $\boxed{\text{always dependent for all } h.}$

15. There are more vectors than there are entries in each vector, so the vectors are $\boxed{\text{not linearly independent}}$

17. The second vector is $\vec{0}$, so the vectors are $\boxed{\text{not linearly independent}}$.

21a. $\boxed{\text{False}}$. Let $A=[1]$. Then $A\cdot\vec{0}=\vec{0}$, but the columns of A are linearly independent. (The rule is that the columns of A are linearly independent only when $A\vec{x}=\vec{0}$ has the trivial solution as its <u>only</u> solution.)

b. $\boxed{\text{False}}$ Let $S=\left\{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}$. Then $\vec{0}$ is in S, so S is linearly dependent. But $\begin{bmatrix}1\\0\end{bmatrix}$ is not in Span$\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$, since Span$\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$ is $\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$. (The rule is that S is dependent only when there is <u>one</u> vector in S that is a linear combination of the rest.)

c. $\boxed{\text{True.}}$ There are more columns than entries in each column, so the columns are dependent.

d. $\boxed{\text{True}}$. Since $\{\vec{x},\vec{y},\vec{z}\}$ is dependent, there are numbers a, b, and c, not all zero, with

$$a\vec{x}+b\vec{y}+c\vec{z}=\vec{0}.$$

If $c=0$, then $a\vec{x}+b\vec{y}=\vec{0}$, and one of a and b is nonzero. But we assumed that $\{\vec{x},\vec{y}\}$ is independent, so this can't happen.

So $c\neq 0$, and so we get:

$$\vec{z}=-\frac{a}{c}\vec{x}-\frac{b}{c}\vec{y},$$

so $\vec{z}$ is a linear combination of $\vec{x}$ and $\vec{y}$. (You don't even need $\vec{x}$ and $\vec{y}$ to be independent for this to work.)

22a. $\boxed{\text{True}}$ Two vectors $\vec{v}$ and $\vec{w}$ are linearly dependent if and only if one is a multiple of the other, and this happens if and only if $\vec{v}$ and $\vec{w}$ lie on a line through the origin.

b. $\boxed{\text{False}}$: $\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$ is dependent because it has $\vec{0}$.

c. $\boxed{\text{True}}$: Since $\vec{z}$ is in Span$\{\vec{x},\vec{y}\}$, there are numbers a and b with $\vec{z}=a\vec{x}+b\vec{y}$. Then $a\vec{x}+b\vec{y}+(-1)\vec{z}=\vec{0}$, and since $-1\neq 0$, $\vec{x}$, $\vec{y}$, and $\vec{z}$ are dependent.

d. $\boxed{\text{False.}}$ $\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$ is dependent, since it contains $\vec{0}$, but there are fewer vectors than entries in each vector.

Jason Ferguson

33. $\boxed{\text{True}}$. Since $\vec{v_3} = 2\vec{v_1} + \vec{v_2}$, we have:

$$2\vec{v_1} + 1\vec{v_2} + (-1)\vec{v_3} + 0\vec{v_4} = \vec{0}.$$

Since 2, 1, and -1 are nonzero, this shows that $\vec{v_1}, \vec{v_2}, \vec{v_3}$, and $\vec{v_4}$ are linearly dependent.

34. $\boxed{\text{True}}$. If $\vec{v_3} = \vec{0}$, then

$$0\vec{v_1} + 0\vec{v_2} + 1\vec{v_3} + 0\vec{v_4} = \vec{0},$$

and since 1 is not zero, $\vec{v_1}, \vec{v_2}, \vec{v_3}$ and $\vec{v_4}$ are dependent.

35. $\boxed{\text{False}}$ Let:

$$\vec{v_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0} \quad \text{and} \quad \vec{v_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then $\vec{v_2}$ is not a scalar multiple of $\vec{v_1}$, since the only scalar multiple of $\vec{v_1}$ is $\vec{0}$.
But since $\vec{v_1} = \vec{0}$, $\{\vec{v_1}, \vec{v_2}\}$ is not linearly independent.

36. $\boxed{\text{False}}$ Let:

$$\vec{v_1} = \vec{v_2} = \vec{v_4} = \vec{0} \qquad \text{and } \vec{v_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then the only linear combination of $\vec{v_1}, \vec{v_2}$, and $\vec{v_4}$ is $\vec{0}$, so $\vec{v_3}$ is not a linear combination of $\vec{v_1}, \vec{v_2}$, and $\vec{v_4}$. But since $\vec{v_1} = \vec{0}$, the set $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ is not linearly independent. (The rule is S is linearly dependent only when one of the vectors in S is a linear combination of the rest.)

Section 1.8

1. $T(\vec{u}) = A\vec{u} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = 1\begin{bmatrix} 2 \\ 0 \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -6 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ -6 \end{bmatrix}}$

$T(\vec{v}) = A\vec{v} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 2 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} 2a \\ 2b \end{bmatrix}}$

3. Need to solve $A\vec{x} = \vec{b}$.

$$\begin{bmatrix} 1 & 0 & -2 & -1 \\ -2 & 1 & 6 & 7 \\ 3 & -2 & -5 & -3 \end{bmatrix} \xrightarrow[\text{Add } (-3)\times(\text{Row 1}) \text{ to Row 3}]{\text{Add } 2\times(\text{Row 1}) \text{ to Row 2}} \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & -2 & 1 & 0 \end{bmatrix} \xrightarrow[\text{to Row 3}]{\text{Add } 2\times(\text{Row 2})} \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 5 & 10 \end{bmatrix}$$

$$\xrightarrow{\text{Recopied}} \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 5 & 10 \end{bmatrix} \xrightarrow[\text{by } \frac{1}{5}]{\text{Multiply Row3}} \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[\text{Add}(-2)\times(\text{Row}3)\text{ to Row}2]{\text{Add }2\times(\text{Row}3)\text{ to Row}1} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

A vector $\vec{x}$ for which $T(\vec{x}) = \vec{b}$ is $\boxed{\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}}$, and $\vec{x}$ $\boxed{\text{is unique}}$

9. Need to solve $A\vec{x} = \vec{0}$:

$$\begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 2 & -6 & 6 & -4 & 0 \end{bmatrix} \xrightarrow[\text{to Row3}]{\text{Add }(-2)\times\text{Row1}} \begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 2 & -8 & 6 & 0 \end{bmatrix}$$

$$\xrightarrow[\text{to Row3}]{\text{Add }(-2)\times\text{Row2}} \begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[\text{to Row1}]{\text{Add }4\times(\text{Row2})} \begin{bmatrix} 1 & 0 & -9 & 7 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9x_3 - 7x_4 \\ 4x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \end{bmatrix}$.

The vectors $\vec{x}$ are those of the form

$$\boxed{\begin{bmatrix} 9x_3 - 7x_4 \\ 4x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} \quad \text{for any numbers } x_3 \text{ and } x_4}$$

In other words $\boxed{\text{the vectors } \vec{x} \text{ that are in } \operatorname{Span}\left\{ \begin{bmatrix} 9 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}}$

11. Need to see if there is an $\vec{x}$ in $\mathbb{R}^4$ with $A\vec{x} = \vec{b}$; i.e. if $A\vec{x} = \vec{b}$ is consistent.

$$\begin{bmatrix} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 2 & -6 & 6 & -4 & 0 \end{bmatrix} \xrightarrow[\text{to Row3}]{\text{Add }(-2)\times\text{Row1}} \begin{bmatrix} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 0 & 2 & -8 & 6 & 2 \end{bmatrix}$$

$$\xrightarrow[\text{to Row3}]{\text{Add }(-2)\times(\text{Row2})} \begin{bmatrix} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This augmented matrix is consistent, so there is an $\vec{x}$ with $A\vec{x} = \vec{b}$, so $\boxed{\vec{b} \text{ is in the range of } \vec{x} \mapsto A\vec{x}}$

Jason Ferguson

17. $T(3\vec{u}) = 3\,T(\vec{u}) = 3\cdot\begin{bmatrix}2\\1\end{bmatrix} = \boxed{\begin{bmatrix}6\\3\end{bmatrix}}$.

$T(2\vec{v}) = 2\,T(\vec{v}) = 2\cdot\begin{bmatrix}-1\\3\end{bmatrix} = \boxed{\begin{bmatrix}-2\\6\end{bmatrix}}$.

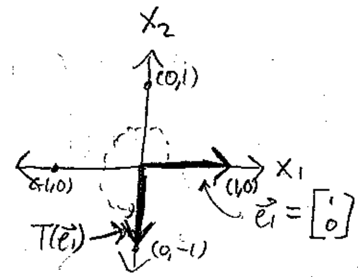
$T(3\vec{u}+2\vec{v}) = T(3\vec{u}) + T(2\vec{v}) = \begin{bmatrix}6\\3\end{bmatrix} + \begin{bmatrix}-2\\6\end{bmatrix} = \boxed{\begin{bmatrix}4\\9\end{bmatrix}}$.

Section 1.9

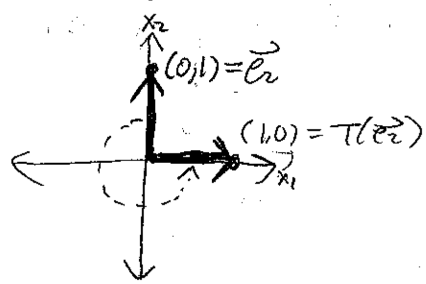
1. The standard matrix is

$$[T(\vec{e}_1)\quad T(\vec{e}_2)] = \boxed{\begin{bmatrix}3 & -5\\ 1 & 2\\ 3 & 0\\ 1 & 0\end{bmatrix}}.$$

3. $T(\vec{e}_1) = T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\-1\end{bmatrix}$, as in the picture to the right



$T(\vec{e}_2) = T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}$, as in the picture to the right.



Matrix is $[T(\vec{e}_1)\quad T(\vec{e}_2)] = \boxed{\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}}$.

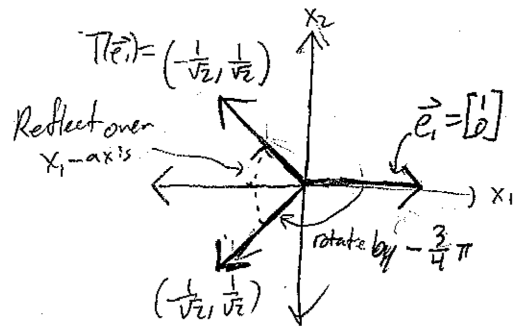
5. $T(\vec{e}_1) = \vec{e}_1 - 2\vec{e}_2 = \begin{bmatrix}1\\0\end{bmatrix} - 2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix}$.

$T(\vec{e}_2) = \vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$.

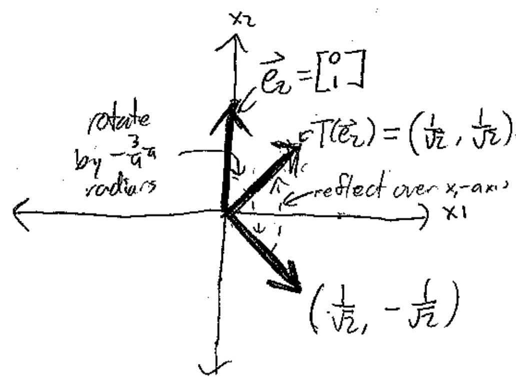
Matrix is $[T(\vec{e}_1)\quad T(\vec{e}_2)] = \boxed{\begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}}$.

Jason Ferguson

7. $T(\vec{e}_1) = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, as in the diagram to the right.



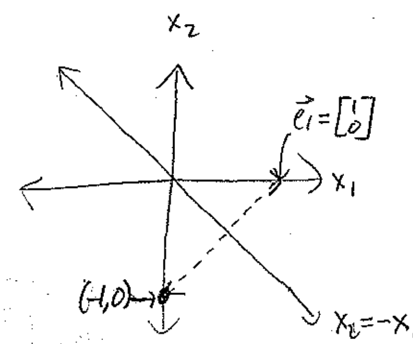
$T(\vec{e}_2) = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, as in the diagram to the right.



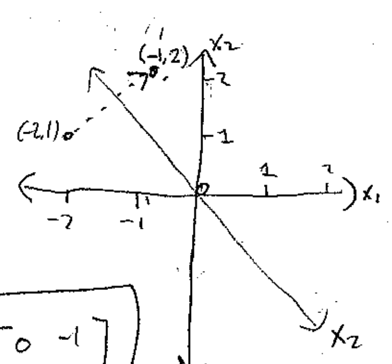
So the matrix of the transformation is:

$$\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \boxed{\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$$

9. $T(\vec{e}_1)$ = reflection of $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ over the line $x_2 = -x_1$
$= \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ as in the diagram to the right



$T(\vec{e}_2)$ = reflection of $\vec{e}_2 - 2\vec{e}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ over the line $x_2 = -x_1$
$= \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ as in the diagram to the right.

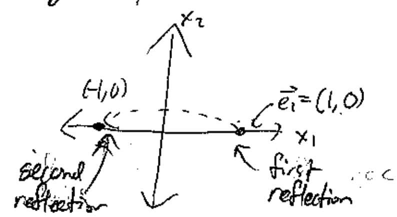


So the matrix of the transformation is $\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \boxed{\begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}}$.

Jason Ferguson

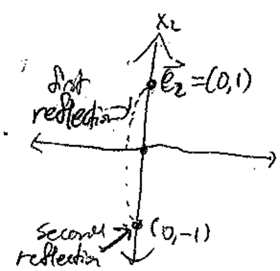
11. Reflecting $\vec{e}_1$ over the x_1-axis gives $\vec{e}_1$, so

$T(\vec{e}_1)$ = point you get by reflecting $\vec{e}_1$ over the x_1-axis, then reflecting that point over the x_2-axis.
= point you get by reflecting $\vec{e}_1$ over the x_2-axis
$= \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

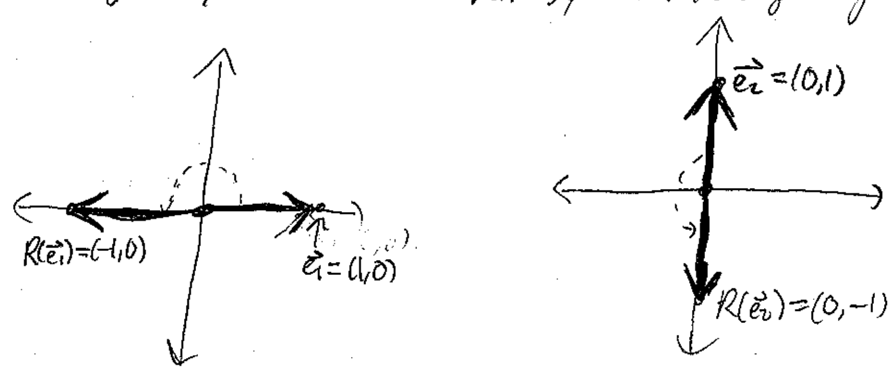


Reflecting $\vec{e}_2$ over the x_1-axis gives $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$, so

$T(\vec{e}_2)$ = point you get by reflecting $\vec{e}_2$ over the x_1-axis, then reflecting that point over the x_2-axis.
= point you get by reflecting $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ over the x_2-axis
$= \begin{bmatrix} 0 \\ -1 \end{bmatrix}$



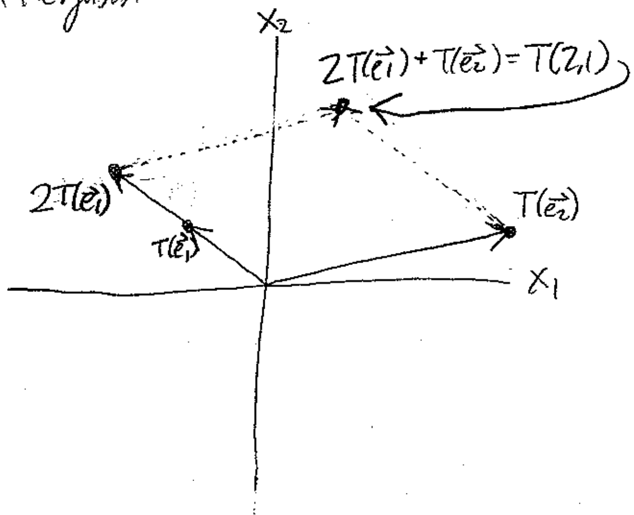
Let $R: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation "rotate counterclockwise about the origin by π radians." Then by the following diagrams



$R(\vec{e}_1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $R(\vec{e}_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Since linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ are completely determined by their effect on $\vec{e}_1$ and $\vec{e}_2$, and $R(\vec{e}_1) = T(\vec{e}_1)$ and $R(\vec{e}_2) = T(\vec{e}_2)$, $R = T$. Therefore T is a rotation by $\boxed{\pi \text{ radians}}$ counterclockwise around the origin.

Jason Ferguson

13.



$$\begin{aligned} T(2,1) &= T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) \\ &= T\left(2\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\right) \\ &= 2T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)+T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \\ &= 2T(\vec{e_1})+T(\vec{e_2}) \end{aligned}$$

15. Suppose the entries of the matrix are:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Then we want

$$\begin{bmatrix} ax_1+bx_2+cx_3 \\ dx_1+ex_2+fx_3 \\ gx_1+hx_2+ix_3 \end{bmatrix} = \begin{bmatrix} 3x_1-2x_3 \\ 4x_1 \\ x_1-x_2+x_3 \end{bmatrix}$$

for all numbers x_1, x_2, and x_3. Plugging in $x_1=1$, $x_2=0$, and $x_3=0$ gives

$$\begin{bmatrix} a \\ d \\ g \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}.$$

Plugging in $x_1=0$, $x_2=1$, and $x_3=0$ gives:

$$\begin{bmatrix} b \\ e \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Plugging in $x_1=0$, $x_2=0$, and $x_3=1$ gives:

$$\begin{bmatrix} c \\ f \\ i \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

So the matrix is $\boxed{\begin{bmatrix} 3 & 0 & -2 \\ 4 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}}$

Jason Ferguson

17. If T were linear and were represented by a matrix ~~A~~, then the columns of A would have to be:

$$T(\vec{e_1}) = T(1,0,0,0) = (0,1,0,0)$$
$$T(\vec{e_2}) = T(0,1,0,0) = (0,1,1,0)$$
$$T(\vec{e_3}) = T(0,0,1,0) = (0,0,1,1).$$
$$T(\vec{e_4}) = T(0,0,0,1) = (0,0,0,1).$$

In other words, A would have to be:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

To check this works:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{bmatrix} = T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right).$$

Therefore, T is represented by the matrix $\boxed{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}}$, so T is linear.

23(a). $\boxed{\text{True.}}$ Suppose T and L are both linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$, and suppose that $T(\vec{e_1}) = L(\vec{e_1})$, ..., $T(\vec{e_n}) = L(\vec{e_n})$.

Then let $\vec{x}$ be any vector in $\mathbb{R}^n$. Then:

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = x_1 \vec{e_1} + \dots + x_n \vec{e_n}.$$

Then $T(\vec{x}) = x_1 T(\vec{e_1}) + \dots + x_n T(\vec{e_n}) = x_1 L(\vec{e_1}) + \dots + x_n L(\vec{e_n}) = L(\vec{x})$,

So $T(\vec{x}) = L(\vec{x})$ for all $\vec{x}$ in $\mathbb{R}^n$, so T and L are the same function.

b. $\boxed{\text{True.}}$ See Example 5 of Section 1.8 for an explanation.

c. $\boxed{\text{False}}$ Any combination of two linear transformations is a linear transformation.

Let $L: \mathbb{R}^m \to \mathbb{R}^n$ and $T: \mathbb{R}^n \longrightarrow \mathbb{R}^p$ be any linear transformations.
Then for all $\vec{x}, \vec{y} \in \mathbb{R}^m$ and numbers c,

$$T(L(\vec{x}+\vec{y})) = T\big(L(\vec{x})+L(\vec{y})\big) = T(L(\vec{x})) + T(L(\vec{y}))$$

and

$$T(L(c\vec{x})) = T\big(cL(\vec{x})\big) = c\,T(L(\vec{x})).$$

So $\vec{x} \longmapsto T(L(\vec{x}))$ is linear.

d. $\boxed{\text{False}}$ Any function $T: \mathbb{R}^n \to \mathbb{R}^m$ sends every vector in $\vec{x}$ to some vector in $\mathbb{R}^m$.
"Onto" means that for each $\vec{y}$ in $\mathbb{R}^m$, there is some $\vec{x}$ in $\mathbb{R}^n$ so that $T(\vec{x}) = \vec{y}$.

For example, $T: \mathbb{R} \to \mathbb{R}$ given by $T(x) = 0$ for all x in $\mathbb{R}$ is a function that is not onto.

e. $\boxed{\text{False}}$ Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then since A has a pivot position in every column, the columns of A are linearly independent, so $\vec{x} \mapsto A\vec{x}$ is one-to-one, even though A is 3×2.

24a. $\boxed{\text{False}}$ See Theorem 10 on p. 85.

b $\boxed{\text{True}}$ See Theorem 10 on p. 85.

c. $\boxed{\text{True}}$ See the first, second, and last rows of the table on p. 87.

d. $\boxed{\text{True}}$ (Assuming the statement means "$T: \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if for each $\vec{x}$ in $\mathbb{R}^n$, $\vec{x}$ is the only vector in $\mathbb{R}^n$ that T sends to $T(\vec{x})$.")
See the discussion at the top of p. 90.

e. $\boxed{\text{True}}$ The transformation $\vec{x} \mapsto A\vec{x}$ is onto only when A has a pivot position in every row. But no matrix can have more pivot positions than columns, so A can't have 3 pivot positions, so A can't have pivot positions in every row.